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ISSUES 3&4

THE ASSOCIATION OF MATHEMATICS TEACHERS OF INDIA

The Association of Mathematics Teachers of India (AMTI) was started in 1965 for the promotion of efforts to improve Mathematics education at all levels. A major aim of the Association is to assist practising teachers of Mathematics in schools in improving their expertise and professional skills. Another important aim is to spot out and foster Mathematical talents in the young. The Association also seeks to disseminate new trends in Mathematics education among parents and public. Other activities of the Association include consultancy services to schools in equipping the Mathematics section of their libraries, in organising children's Mathematics clubs and fairs, in setting up teacher centres in schools, in conducting Mathematics laboratory programmes, in holding practical tests in Mathematics in assisting children in participating investigational projects.

The Association holds " National Mathematical Talent Search Competition " annually and organizes Orientation Courses, Seminars and Workshops for teachers and courses for talented students. A national conference is held annually in different parts of the country for teachers to meet and deliberate on important issues of Mathematics education. Innovative teacher award has been instituted to give public recognition to enterprising and pioneering teachers of Mathematics for which entries from teachers are invited.

An award for contributions to the Mathematics Teacher relating to History of Mathematics in the context of mathematics education had been instituted by Prof.R.C.Gupta.

"The Mathematics Teacher (India)" (MT) is the official quarterly journal of the Association and is issued twice a year. It has been approved for use in schools and colleges of education by the Government departments of education in many States. Besides MT the Association also brings out Junior Mathematician (JM), three issues in a year, especially for school students in English and Tamil.

The membership of the Association is open to all those interested in Mathematics and Mathematics Education. The membership fee inclusive of subscription for "**The Mathematics Teacher (India)**" and effective from April 1993 is as follows:

Subscription for India*

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Life	Rs. 500	Rs. 1000
Annual (Ordinary)	Rs. 50	—
Junior Mathematician - Life	Rs. 250	Rs. 250
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The Journal "The Mathematics Teacher" will be supplied free to all members.
Fifty or more subscriptions to Junior Mathematician will entail 20% discount.

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Dr. S.Muralidharan
Editor

EDITORIAL

In this issue, we are happy to publish the text of the R.C.Gupta Endowment lecture delivered by Professor E Krishnan and that of P.L.Bhatnagar Memorial Endowment lecture delivered by Professor Ambat Vijayakumar during the 48th Annual Conference of our Association.

R Swaminathan's thought provoking article on the need of a practical approach in Undergraduate Mathematics curriculum planning and teaching will make many of us agree with his suggestions and do our bit towards improving the quality of Mathematics teaching at all levels.

Most of us have been trained extensively on Geometry during our school days. The study of theorems and solving of riders taught us the essentials of sequential logical thinking and improved our problem solving abilities. Unfortunately, the current day school curriculum gives Geometry a less prominent place. The arguments against Geometry are plenty and Subramanyam Durbha argues forcefully for more emphasis on Euclidean Geometry in schools. His article along with the dismal showing reported on the achievement level of students

by Utpal Borah should be an eye-opener for the Mathematics teachers in India.

Archimedes is credited with the computation of area of a parabola and he used a technique that later became the basis of integration. Professor Mamikon A.Mnatsakanian developed the sweeping tangent technique to compute areas of more complex curves. His recent book with Tom A.Apostol, "New Horizons in Geometry" is a must read for all Geometry lovers. S Saroja in her article "On Mamikon's Theorem" presents an elementary proof of Mamikon's theorem for ovals in the plane and also general plane curves. This proof is more accessible to those who have basic knowledge of differential geometry.

Polya proved an interesting theorem on symmetric random walks. In dimensions higher than 3, the symmetric random walk is transient (that is, the probability of returning to the starting point is not equal to unity). S Muralidharan presents an elementary proof of this theorem.

Intuition plays a major role in Mathematics. L.R.Ganesan presents an intuitive approach to product of Gauss functions.

Finally, we present an article on derivation of closed form for sum of terms of an arithmetic progression in terms of Stirling numbers of the second kind by Faiz Imam.

LaTeX has revolutionized the Mathematical typesetting. We want to request readers who submit articles for publication in Mathematics Teacher to submit manuscripts prepared with LaTeX. This not only reduces the cost of typesetting but also ensures timely publication of the journal. For learning LaTeX, Professor S Parthasarathy has recently released an e-book "Let's Learn LaTeX". This free ebook on LaTeX is a sequel to his article "When Grace Meets Beauty – LaTeX Meets Mathematics" published in this journal (Vol 46, No 3-4, pp 141-49). The download instructions are available from: http://www.freewebs.com/profpartha/teachlatex.htm.

Learning $\LaTeX$ is a lot of fun and I would like to encourage all the readers to attempt preparing manuscripts using this wonderful typesetting system.

Several readers (Ms Prema, Professor Kameswara Rao and others) have pointed out the errors in the Mathematics Teacher issue Vol 50 (1–2). The answer to the question 3, in Gauss Contest 2014, must be 17 and the question 13 is ambiguous. The errors are regretted.

CONTENTS

Trigonometry — History and Pedagogy

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1 Introduction

Mathematics originated in measurement. Numbers were invented to denote the results of these measurements and arithmetical operations on numbers were defined to express the various ways of combining measurements. In this sense, mathematics is the science of quantification. This was the prevalent view of mathematics, even as late as the eighteenth century, as can be seen from Euler's comment:

> *Mathematics, in general, is the science of quantity; or, the science which investigates the means of measuring quantity*

But even as early as the third millennium BC, we see an interest in pure numbers and abstract shapes, perhaps as a result of division of labor, with mathematicians emerging as a separate working class.

Thus we see the evolution of mathematics following a dual course, as a practical science applied to the measurements of physical objects and their inter-relationships, and also as a purely aesthetic pursuit devoted to the study of ideal shapes and pure numbers solely based on logic. But practical

Prof. R.C. Gupta Endowment Lecture on History of Maths Delivered in 48th Conference.

considerations often lead to abstract concepts and such theories sometimes find unexpected applications far removed from the source.

Unfortunately, the teaching of mathematics at all levels ignore this dialectics most of the time, so that it becomes a mechanical application of computational procedures and meaningless manipulation of symbols. Incorporating the history of a mathematical concept will definitely give some motivation to students, by first giving the physical context in which the idea originated, then its later evolution and finally its current state, both as an abstract notion and as a practical tool. Such a scheme also serves to establish connections between different parts of mathematics and connections between mathematics and other subjects as well.

In this talk, I illustrate these ideas with reference to the subject of trigonometry.

2 History of trigonometry

According to the current availability of historical documents, the first occurrence of what maybe termed a trigonometric idea occurs in the famous A'hmose papyrus of Egypt, dated around 1650 BC. One of the problems in it is this:

> *If a pyramid is* 250 *cubits high and the base is* 360 *cubits long, what is its* seked*?*

To understand the meaning of *seked*, we must look at the solution given in the papyrus:

> *Take half of* 360*; it makes* 180. *Multiply* 250 *to*

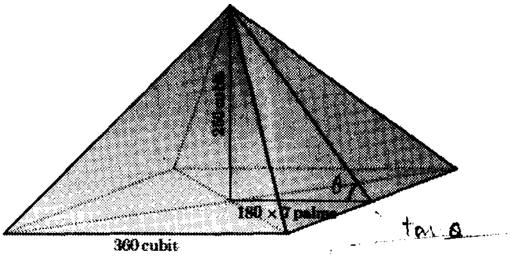

Figure 1

> *get* 180 ; *it makes* $\frac{18}{25}$. *A cubit is* 7 *palms. Multiply* $\frac{18}{25}$ *by* 7 *to get* $5\frac{1}{25}$. *It is the* seked

So, what is computed as *seked* is the number of times the height, measured in cubits, is half the base of the pyramid, measured in palms. The figure 1 will help to make matters clear. So, *seked* is a measure of the slant of the pyramid, measured against the vertical. There is a reason for computing the slope like this.
In constructing a giant Egyptian pyramid, a step pyramid was first built using rectangular blocks of stone and then the sides were filled up with trapezoidal stones to make smoothly sloping triangles on all sides (see Figure 2). For the sides to be smooth, the ratio of the height of each such casing stone to the deviation of the top corner from the vertical should be the same as the ratio of the pyramid's height to half its base length (Figure 3): Thus the Egyptian *seked* is equivalent to what we now call the cotangent of an angle. However. It must be noted that instead of considering this as a function of the measure of the angle as we now do, this was *the* measure of the angle. Architects and builders to this day use the the ratio of *rise* and *run* as the measure of the angle (Figure 4). The pyramid builders were well aware of the fact that even though rise and run change

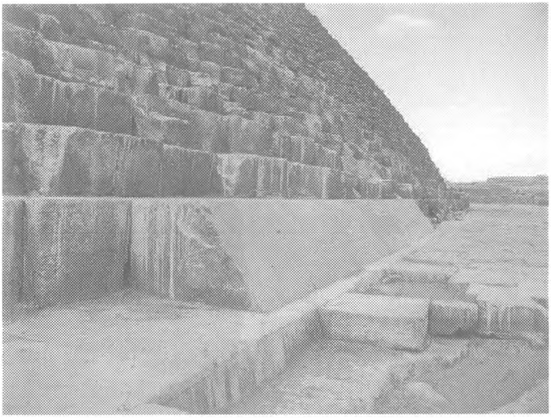

Figure 2

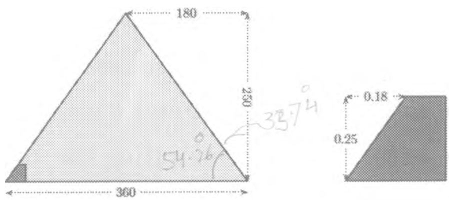

Figure 3

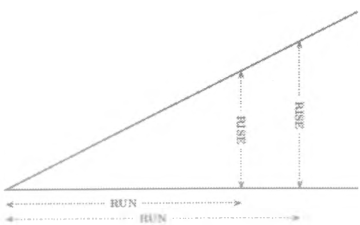

Figure 4

individually with position, their ratio remains constant for a particular angle.

Thus the trigonometric functions cotangent and tangent arose from earth-bound activities such as the construction of buildings; but the sine and cosine functions came from the sky—more precisely from astronomy.

In most of the ancient civilizations based on agriculture, astronomy was an important subject of study, since accurate prediction of seasons and hence effective agriculture demands it. For sky-watchers, the measure of an angle is an amount of rotation, rather than a measure of inclination as for builders on the earth. They realized that for various circles drawn with the center on the vertex of an angle, the actual length of the arcs intercepted by the angle vary, but the ratio of the length of an arc to the circumference of the circle remains a constant (Figure 5).

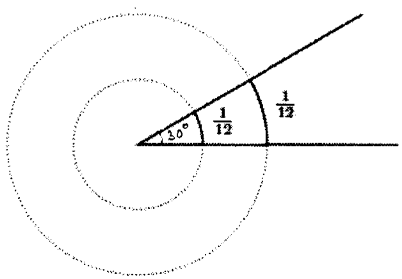

Figure 5

Ancient Babylonians often expressed the measure of a rotation as this fraction multiplied by 360. Thus the amount of rotation of the upper line in the figure above from the horizontal would be taken as $\frac{1}{12} \times 360 = 30$. (Why the Babylonians chose 360 is a moot point; some historians says it is because

they miscalculated the number of days in a year as 360 days, while some others say that it is because their number system was based on 60.) Anyway, this is the basis of our current reckoning of angles in degrees. Incidentally, it was much later in the eighteenth century that the English mathematician Roger Cotes introduced the idea of measuring an angle as the ratio of the intercepted arc and its radius (instead of the circumference of the whole circle); this measure was named *radian* by the English physicist James Thomson in the nineteenth century.

In studying the orbits of planets, ancient astronomers often had to compute the length of the chord of a circle in terms of the length of the subtending arc. In the first century BC, Hipparchus in Greece computed such a table of chords for a circle of fixed radius, for angles with increments of $7\frac{1}{2}^\circ$. This was refined by Claudius Ptolemy of Egypt to angles with an increment of $\frac{1}{2}^\circ$, in the second century CE.

Using such a table of chords, the Greek mathematicians could find the lengths of the perpendicular sides of any right triangle, given the hypotenuse and any other angle. For example, consider this problem:

> *The hypotenuse of a right-angled triangle is* 6 *and one of its angles is* 20° *; to compute the lengths of the other two sides*

To do this, we consider the triangle together with its circumcircle and draw the radius to the right angle (Figure 6)

From the table, it is found that the length of a chord of central angle 40° is 0.684 of the radius and from this, one side of the

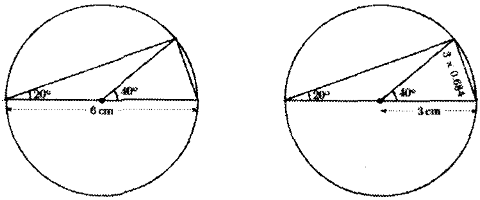

Figure 6

triangle can be computed as $3 \times 0.684 = 2.052\,cm$.

In the above computation, we first double the angle ($20°$ to $40°$), find its chord from the table (0.684) and then multiply the hypotenuse by half this number ($6 \times \frac{1}{2} \times 0.684$). Some mathematicians tried to reduce this computation, by making a table associating each angle with half the chord of double the angle (Figure 7). We see such a table in *suryasiddhanta*,

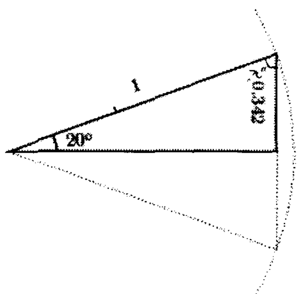

Figure 7

written in India in the 5th century CE, where this measure is called the *ardhjya* of the arc. It is nothing but the sine of the angle.

The etymology of the word *sine* is interesting. Later works in Indian astronomy refer to this simply as *jya*, instead of the original *ardhjya*. (The word *jya* in Sanskrit literally means "bow-string".) When these works were translated into Arabic during the eight century, this word was transliterated as *jiba* and was writen *jb*, since Arabic is written without short vowels. When these Arabic works were translated into Latin, this was mistook for the arabic word *jaib*, which means "bay"or "fold" and the Latin equivalent *sinus* was used for this word. The English form *sine* was introduced in the sixteenth century.

As an illustration of the applications of these ideas, let us see how the Persian scholar Abu Rayhan al-Biruni computed the radius of the earth in the tenth century. He first measured the angle between the the top of a mountain and the plain at two points at sea level with a known distance apart, using an inclinometer (Figure 8).

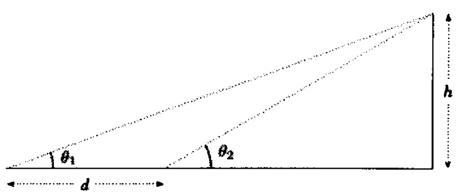

Figure 8

and calculated the height of the mountain using the trigonometric formula:

$$h = \frac{d \tan \theta_1 \tan \theta_2}{\tan \theta_2 - \tan \theta_1}$$

He then climbed to the top of the mountain and measured the dip angle, and used this and the height of the mountain found

earlier to compute the radius of the earth (Figure 9)

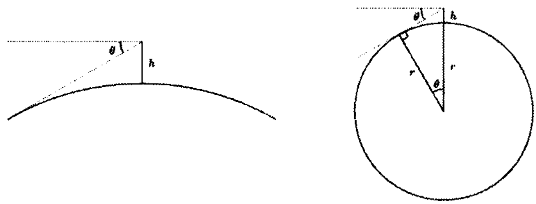

Figure 9

$$r = \frac{h \cos \theta}{1 - \cos \theta}$$

His estimate of 6,339.9 km (in todays units) for the Earth radius was only 16.8 km less than the modern value of 6,356.7 km

Ptolemy computed the chords using various theorems and techniques of pure geometry (see Chapter 4 of [1]). But in the fourteenth century Madhavan of Keralam, India came up with a purely analytic method to compute the sine:

> *Multiply the square of the arc by the the radius and take the result of repeating. Divide by the square of the successive even numbers increased by that number and multiplied by the square of the radius. Place the successive results so obtained one below the other and subtract each from the one above. These together give the jya*

To explain this in modern terms, let's take an arc of length s in a circle of radius r (Figure 10)

Then the *jya*, as explained earlier, is $\frac{1}{2}c$ (half the chord of double the arc). Madhavan's method consists of the following steps:

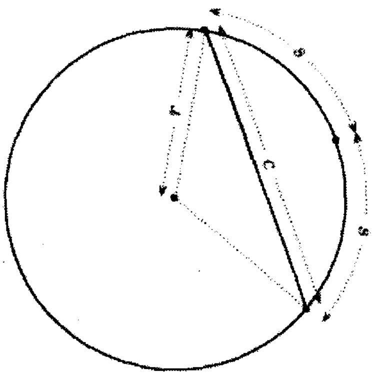

Figure 10

(i)

1. compute the numbers, $s \times s^2 = s^3$, $s^3 \times s^2 = s^5$ and so on
2. compute the numbers, $(2^2+2)r^2$, $(2^2+2)(4^2+4)r^4$ and so on
3. compute the quotients, $\dfrac{s^3}{(2^2+2)r^2}$, $\dfrac{s^5}{(2^2+2)(4^2+4)r^4}$ and so on

The final computation, according to Madhavan's instructions is

$$\begin{aligned}\tfrac{1}{2}c &= s - \left(\frac{s^3}{(2^2+2)r^2} - \left(\frac{s^5}{(2^2+2)(4^2+4)r^4} - \right.\right. \\ &\qquad \left.\left.\left(\frac{s^7}{(2^2+2)(4^2+4)(6^2+6)r^6} - \cdots\right)\right)\right)\end{aligned}$$

If we note that

$$2^2 + 2 = 2(2+1) = 2 \times 3 = 3!$$
$$(2^2+2)(4^2+4) = 3!4 \times (4+1) = 5!$$

and so on, then we have

$$\tfrac{1}{2}c = s - \frac{s^3}{3!r^2} + \frac{s^5}{5!r^4} - \frac{s^7}{7!r^6} + \cdots$$

Again, taking the central angle of the arc as x radians, we have $s = rx$ and $\frac{1}{2}c$ as $\sin x$. Using these in the above equation and simplifying, we get

$$\sin x = x - \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 - \tfrac{1}{7!}x^7 + \cdots$$

We don't know how Madhavan arrived at this method, since neither he nor his disciples give any proof. Three centuries later, Newton and Leibnitz rediscovered this method using the tools of calculus they had developed.

Apart from providing a practical method to compute sines, the series representation had another consequence. The geometrical definition of $\sin x$ in terms of a circle (or triangle) has meaning only for $0 < x < \frac{1}{2}$; but the series above converges for all real values of x. Thus using this series, sine can be extended to a real-valued function on the set of all real numbers, transcending its geometric origin.

The invention of analytic geometry by Descartes and Fermat in the seventeenth century gave a method to define this extended sine function geometrically: given a real number t, draw an arc with origin as the center and the unit of measurement as radius, starting at $(1, 0)$ and of length t; the y-coordinate of the end of the arc is $\sin t$ and the x-coordinate is $\cos t$ (Figure 11)

Analytical geometry also gave back a geometric shape to the sine function in a new form (Figure 12) The wave nature of the sine function makes it a convenient tool to express periodic

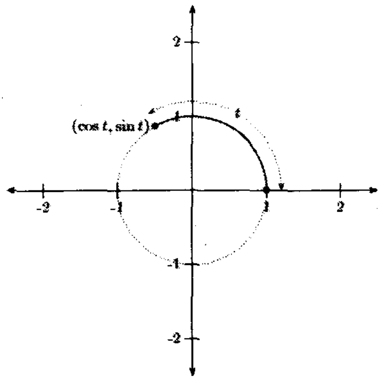

Figure 11

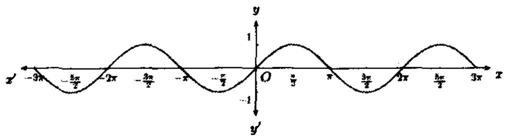

Figure 12

phenomena in physics, such as simple harmonic motion. Again, the studies on vibrating strings and transmission of heat in the nineteenth century led to the theory of approximating all periodic functions in terms of the sine and cosine series and to Fourier Series; and this now has applications ranging from comparison of DNA sequences in biology to data compression in computers.

3 Teaching trigonometry

The evolution of a mathematical concept is not linear, with diverse threads interweaving, so that we cannot hope to teach

the concept strictly adhering to history. But then with hindsight, we can discern a linear strand to base our teaching on, with the other strands serving as interesting explorations. The teaching of the subject should emphasize the dual nature of mathematics as described in the first section. Here I indicate an outline of such a course in trigonometry.

At some point in high school geometry, we talk about congruency of triangles. A historical motivation for this idea can be given by a tale of Thales, a Greek scholar of the sixth century BC. It is told that he was asked by the king to measure the distance of a ship, moored at sea, from the shore. This is how he is said to have done it. He first erected a pole at water's edge directly in line with the ship. He erected another pole on the shore some distance away from the first and finally a third pole exactly at the middle of the first two. Then he walked back from the second pole, perpendicular to the shore and keeping the ship in sight. He marked the position where this pole came into the line of sight with the ship (Figure 13).

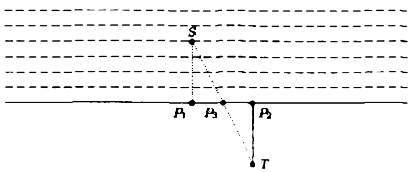

Figure 13

In the figure above, S is the ship, P_1, P_2, P_3 are the poles in order and T is the final position of Thales. Thus Thales ingeniously flips the triangle SP_1P_3 at sea onto the triangle TP_2P_3 on land, so that he can measure SP_1 as TP_2.

The fact that triangles with sides of same length have angles of the same size leads to the theoretical question whether the converse is true. On seeing this is not so, the problem is to see what the relation between the sides is in this case. The idea that right triangles with the same angles have sides of the same ratio can be illustrated again through the tale of another exploit of Thales. It is said that Thales calculated the height of an Egyptian pyramid by measuring its shadow against the show of his staff. This is how Plutarch, a Greek historian of the first century CE recounts the tale:

> *without trouble or the assistance of any instrument merely set up a stick at the extremity of the shadow cast by the pyramid and, having thus made two triangles by the impact of the sun's rays, ... showed that the pyramid has to the stick the same ratio which the shadow has to the shadow*

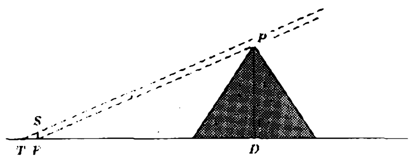

Figure 14

This means that in the Figure 14, $PD : SF = DT : TF$ and so

$$PD = \frac{DT}{TF} \times SF$$

A discussion on this leads to the idea that any triangles with the same three angles have sides of the same ratio and thus to the idea of similarity.

The facts that the sides of a triangle determine its angles, and

angles of a triangle determine the ratio of its sides leads to the theoretical question of actually determining these. This leads to trigonometry.

We can start with some simples cases, such as determining the ratios of sides of triangles of angles $45°, 45°, 90°$ and $30°, 60°, 90°$. We can even use some ideas of Ptolemy mentioned in the last section to get the ratios of sides of a triangle with angles $15°, 75°, 90°$. We can then tell the students that such ratios are computed for all right triangles through centuries of efforts and now readily available as tables. The historical origin of tan and cot as computation of slope, and that of sine as computation of chords must be emphasized to show how diverse ideas are unified into a single theory.

After introducing the various trigonometric ratios, it can be shown that the angles are determined by the sides according to the rules

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \quad \cos B = \frac{c^2 + a^2 - b^2}{2ca} \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

and that the angles determine the ratio of the sides as

$$a : b : c = \sin A : \sin B : \sin C$$

Along with this, some applications of elementary trigonometry can also be introduced. Past applications such as al-Biruni's computation of the radius of the earth, as described in the last section and also modern applications such as calculating the height of lunar mountains from satellite photographs showing their shadows must be discussed. A project based on this can be found at `http://stupendous.rit.edu/classes/phys236/moon_mount/moon_mount.html`

The extension of sine as a real function can be discussed in Higher Secondary classes. Both the geometric method using analytic geometry and the analytic method based on Madhavan series should be discussed and their equivalence highlighted. It must be noted that these methods, as described in the last section, treat sin as a function on arc length. This serves as a motivation for introducing the radian measure of an angle. It must be emphasized and illustrated through examples that the degree measure of angles is more convenient in practical applications, while the radian measure is more useful in theoretical developments.

The graph of sine function and its approximation by polynomials using Madhavan series are best discussed using dynamic geometry software, such as GeoGebra. Further developments such as Fourier series and the extensions of the trigonometric functions functions to complex numbers are more suitable for a college course.

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Applications of Mathematics - Excitements and Challenges.

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I am extremely honored to deliver this endowment lecture in memory of Padmabhushan Prof. P.L. Bhatnagar (1912-1976)*, who initiated the Mathematics Olympiad Programme in India. I have been co-ordinating this activity in Kerala since 1990 as the Regional Co-ordinator of Indian National Mathematical Olympiad. Let me first of all, thank the AMTI for choosing me as the Endowment Speaker for its 48th Annual Conference to be held in Cochin during 10-12, January 2014.

Origin of Mathematics may be traced back to at least the Vedic period. Mathematics did not spring into being fully formed. It grew from the cumulative efforts of many people – the Indians, the Greeks, the Babylonians, the Egyptians, the British and so on, from many cultures, who spoke many languages. Once a mathematical discovery has been made, it becomes available for anyone to use and thereby acquires a life of its own. Good mathematical ideas seldom go out of fashion, though their implementation can change. The rise of civilization and the rise of mathematics have gone hand in hand.

Todays society could not function without mathematics. Everything that we use today, the modern gadgets like

P.L. Bhatnagar Memorial Endowment Lecture in the 48th Annual Conference of the Association of Mathematics Teachers of India (AMTI).

television, cell phone etc. and the digital innovations like the e-mail and e-commerce use mathematics in one form or the other. Mars missions and the Chandraayan could not have been possible, without mathematics. In fact, the entire world revolves around this subject.

Numbers and geometrical shapes have been attracting people since time immemorial. However, understanding one of the most fundamental objects in mathematics – prime numbers, remains an enigma. A complete understanding of these numbers may make you a millionaire. In the meeting held on May 24, 2000 to mark the 100th anniversary of David Hilberts challenge in the ICM, Paris when he posed 23 problems belonging to different topics, to the mathematicians all over the world, to be solved in the next century, a fresh set of 7 problems – Millennium Prize Problems were listed, including the much celebrated RIEMANN HYPOTHESIS, the solution of which may fetch you one million dollars! This seems to me, one of the exciting challenges.

Number Theory forms the basis of many important security codes used for e-commerce, thanks to the fundamental problems of communications posed by Claude Shannon (1916-2004) and the inventors of the best known such codes RSA - Ronald Rivest, Adi Shamir and Leonard Adleman. Number Theory has slowly shifted from its once PURE status as G.H. Hardy put it to the most APPLIED! The applications of number fields to the algebraic coding theory leads us to another excitement called the BCH codes named after two Indians R.C. Bose, D.K. Ray Choudhary and the French, Hockengham.

It is known that there are infinitely many primes. But, where

are they? Can we find a formula to generate all the primes? In fact, they seem to occur some what irregularly and there is no simple way to predict the next number on the list. Euclid introduced primes in Book VII of his magnum opus, the ELEMENTS. He gave proofs of three important properties as well. Using the modern terminology, they are :

1. Every number can be expressed as a product of primes – The Fundamental Theorem of Arithmetic.

2. This expression is unique except for the order in which the primes occur.

3. There are infinitely many primes. (what Euclid actually stated and proved was slightly different – He wrote, in Proposition 20, Book IX, that, Prime numbers are more than any assigned multitude of prime numbers).

Interestingly, in 1988 the British mathematics educator David Wells asked the mathematical community to list most beautiful theorems. There was a tie for second place, between the

- Eulers Polyhedral Formula: $V - E + F = 2$ and

- Euclid: There are infinite number of primes.

The first was undoubtedly, the magic equation of Euler,

$$e^{i\pi} + 1 = 0$$

There is NO largest prime. But, the largest *known* prime is of the form $2^p - 1$. Numbers of the form $2^p - 1$ where p

is prime are called MERSENNE PRIMES, after the French mathematician, Marine Mersenne (1588 – 1648). He, in his Cogitata Physica Mathematica (1644), conjectured that these numbers are prime for $p = 2, 3, 5, 7, 13, 17, 19, 31, 67, 127$ and 257 and composite for all other numbers up to 257. But, it was found that his numbers are composite when $p = 67$ and 257 and that there are three more primes with $p = 61, 89, 107$. On January $25th, 2013$, the largest known prime number, $2^{57,885,161} - 1$, was discovered on Great Internet Mersenne Prime Search (GIMPS) — a distributed computing project, by volunteer Curtis Cooper's computer. The new prime number has $17, 425, 170$ digits. Though mathematically not exciting, it tests the power of new super computers.

Recently in 2013, there was a great breakthrough in proving the long standing TWIN PRIME CONJECTURE – there are infinitely many twin primes – pairs of primes with gap 2, achieved by a Chinese Mathematician Prof. Yitang Zhang, at the University of New Hampshire, USA. He proved that there are infinitely many primes with a gap N such that N is less than seventy million. We know many things about primes BUT, still we do not know many things about the primes! Primes and number theory in general continue to pose challenges.

Galileo Galilei (1564-1642) in his book The Assayer (1623) remarked that the book of nature is written in the language of mathematics, and its characters are triangles, circles and other geometric figures, without which it is humanly impossible to understand a single word of it. His successor, Sir Issac Newton (1642– 1727) in his Mathematical Principles of Natural Philosophy had reduced the system of the world to differential

equations and these are deterministic. That is, once the initial state of the system is known, its future is determined uniquely for all time. The growth of scientific determinism assures that simple causes may produce simple effects, implying that complex effects must have complex causes. These observations lead to the theory of chaos during early 1960s. The American Meteorologist Edward Lorenz in 1963 described a model to analyze the difficulty in weather predictions. These led to the popular term Butterfly Effect in which the flapping of a butterflys wings leads, a month later, to a hurricane on the far side of the globe. These mathematical challenges get an added attraction, since UNESCO and many other scientific organizations have declared the year 2013 as the Year of Mathematics for Planet Earth. The objective of this declaration is to enlighten the public on the increasing role of mathematics in day to day life such as climate analysis etc. On an another line of thought, contrary to that of Galileo, the Polish Mathematician Benoit Mandelbrot (1924 - 2010) realized that there is roughness and irregularities in nature and the Galilean thoughts cannot explain completely many phenomena in nature. Some exciting questions like, How long is the coastal line of Britian? What is the shape of the clouds and the mountains?, led him to the Theory of Fractals. Thus, when a pebble is shown, Newton and Mandelbrot see it entirely differently! A new geometry of nature was thus born, which now finds its exciting applications in medical imaging ! Fractal phenomena is found everywhere in nature.

On 23rd October 1852, a school boy named Francis Guthrie, asked an innocent looking question to his teacher De Morgan "While colouring the different counties in my countrys map,

I required four colours when adjacent counties are coloured differently. Does FOUR COLOURS SUFFICE for all such maps?" This question lateron came to be known as the 4 Colour Conjecture in Graph Theory. Neither De Morgan nor his teachers and their teachers, Hamilton, Gauss, etc. could even understand the intricacies of this question, which defied a solution till 1974, when Kenneth Appel, Wolfgang Haken and Koch with the help of computers, gave a proof that 4 colours suffice to color any planar map. The proof technique became historic - the first theorem in the entire mathematics to have been proved using a computer. Though, several equivalent formulations of 4CC are known, an analytical proof is still a distant dream! Hence, this problem also causes an exotic excitement ! In the course of my talk, we shall hear many more such stories – Eulers Problem of thirty six Officers and about the Euler Spoilers, Tower of Hanoi Problem, the Shoe Lace Problem and the $(3n+1)$ problem etc. Each of these will take you to different levels of challenges.

If we feel that these problems are in the mathematics circles since many years, the question, "CAN WE USE MATHEMATICS TO FIND OUR FRIENDS?" is of very recent interest. These lead to problems in Social Network Analysis.

Centuries after these puzzles and problems, its solutions and counter examples, enriching the field of Mathematics, we have now a pertinent question : WHERE DOES MATHEMATICS STAND? TO WHERE DOES MATHEMATICS LEAD US ALL?

To be frank, nobody knows! The internet affects our day to day life. How we store and retrieve information, how we order books

online, how we choose our life partners through this facility are the mathematical challenges that we face now. The web graph, and its study called INTERNET MATHEMATICS is an active field of research. But, new techniques are needed to model and analyze the properties of the world wide web. Computer Science, Probability Theory and Graph Theory etc. are fast converging towards this end.

Our country needs more people to do research in fundamental sciences and mathematics, which could possibly help us understand our world better. I feel that national level organizations like AMTI will take up these issues and propagate the same in all possible forums. I am an OPTIMIST! I conclude this write up with a quote from Albert Einstein, "The mere formulation of a problem is far more essential than its solution, which may be merely a matter of mathematical or experimental skills. To raise new questions, new possibilities, to regard old problems from a new angle requires creative imagination and marks real advances in science."

***About Prof Bhatnagar:**

Prabhu Lal Bhatnagar was born on August 8, 1912 in Kota in Rajasthan, He went to school first at Kota, then Herberter College, Kota in 1934, when he completed his BSc degree at Maharajah's College, Jaipur. Bhatnagar first worked with Professor BN Prasad on the summability of Fourier and Allied Series. His research work at Allahabad from 1937 – 39 included the solution of second order linear ordinary differential equations by the Laplace Transform technique. He obtained his DPhil degree in Mathematics in 1939 for his thesis entitled "On the origin of the solar system" under the supervision of

Professor A C Banerji. In 1951 Bhatnagar went to Harvard University, Cambridge, Massachusetts as a Fulbright scholar.

In 1950 he was elected Fellow of the National Institute of Sciences (now INSA, Indian National Science Academy). In 1955, he was elected fellow of the Indian Academy of Sciences. In January 1956, the Indian Institute of Science, Bangalore invited him to join as the first Professor of the newly created Department of Applied Mathematics. Bhatnagar was made the first Director of the Mehta Research Institute (MRI) of Mathematics and Mathematical Physics at Allahabad in July 1975.

He was among other things, President of the Association of Mathematics Teachers of India (1968-76). He formed the Bangalore Mathematical Association and under its auspices, apart from lectures etc., introduced Mathematics Olympiads on the lines of those held in East European Countries. This was the first ever Mathematics Olympiad held in India. Now, the National Board for Higher Mathematics (NBHM) conducts the Indian National Mathematics Olympiad (INMO) all over India to detect and nurture talent in mathematics among students at the level of standard XI.

For his service to the nation, Prof.Bhatnagar was awarded the Padma Bhushan on January 26, 1968. He passed away on October 5, 1976.

Suggested reading:

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Needed – A Practical Approach towards Undergraduate Mathematics Curriculum Planning and Teaching

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Nowadays, it is becoming increasingly difficult to find young motivated students opting mathematics courses. There are two reasons for this unfortunate trend: i) In many text books, Mathematics is projected as a bundle of symbols, theorems and proofs without any cohesion. ii) The present teaching methods do not address the difficulties faced by the students. While the students must be exposed to the basic courses like algebra, real and complex analysis, vector analysis and differential equations, too much emphasis on rigor and formalism can deter the students from taking up mathematics courses. The curriculum should, therefore, contain a balanced mixture of pure and applied subjects.

If one goes through the history of mathematical development, one can find several facets of mathematics – the aesthetic, the intuitive, the scientific and the recreational. The theory of numbers, for example, calls for a high degree of intuition. The construction of magic squares is an example of recreational mathematics. But a large body of mathematics has been developed based on bare necessities of life and the demands made by the physical sciences. The development of the number systems was based on the day-to-day requirements of trade and domestic management. The development of

geometry was of course dictated by land measurements and architecture. Later developments in Mathematics, particularly during the period 1600-1900, were accompanied by amazing developments in Physics and Engineering. It is noteworthy that great mathematicians belonging to this period like Newton, Gauss, Riemann, Euler, Lagrange, Cauchy, Bernoulli and Fourier were all physicists, astronomers and engineers. They were all working on physical problems which required mathematical tools for analysis. The discovery of calculus, for example, was a natural consequence to the study of planetary motion. Similarly, the developments in complex analysis, vector analysis and differential equations were linked in a natural way to the understanding of the physical laws governing electrostatics, fluid mechanics, heat transfer, elasticity, celestial mechanics and control theory. Problems in Mechanics (like the Brachistochrone) have led to the discovery of interesting geometric shapes like the cycloid. Problems in Mechanics, Fluid Mechanics and Structural dynamics are closely linked to geometrical concepts like curvature, geodesics. Applications often lead us towards the discovery of new results in pure Mathematics and the methods of proof. In short, developments in mathematics and the physical sciences enrich each other.

Later, the discovery of the theory of sets by George Cantor in 1895 brought about a distinct change in the nature of modern mathematics. While set theory has put mathematics on a firm foundation, it has also added an abstract character to the study of mathematics. This has no doubt increased the creativity during the last century. One can cite interesting developments in Topology, Functional Analysis, Algebra and Algebraic Geometry. There is no doubt that these developments have

not only provided deep insight into many unsolved problems in number theory, geometry and algebra, but have opened up new frontiers of applications. Perhaps these topics can be reserved for post graduate courses.

The rapid strides made during the last century in the physical and biological sciences and the growth in technology have made increasing demands on mathematical modelling. The initiative for further development of mathematical models has already been wrested from mathematicians by engineers and technologists. One can already see clear evidences of this trend from the development of computer algorithms, linear and non-linear programming algorithms, and several innovative numerical techniques for solving problems in fluid mechanics, structural engineering and optimal control. While some of these developments make use of the conventional disciplines like linear algebra, differential equations etc, one can also find novel applications of fuzzy sets, group theory, graph theory, game theory, number theory, quaternions, wavelet transforms, theory of fractals and the creation of new algorithms like neural networks and genetic algorithms.

The objective of this article is to draw the attention of the mathematics curriculum planners and text book writers to the above mentioned realities of the modern world. The modern technological advances in computers, space, atomic energy, bio-technology and defence offer a lot of scope for innovative mathematical modeling. Moreover, there are research organizations in these sectors which offer employment opportunities. The curriculum and the text books should reflect these realities. At the undergraduate level, the stress

should be on applications of Mathematics. The only way to attract talented young students towards mathematics courses is to make these courses application and employment oriented. Applications also help in better understanding of the concepts. Apart from the conventional subjects like algebra, analysis, differential equations etc subjects like numerical methods, computer programming, linear and non-linear programming, graph theory should also be taught. Topics like Mechanics, Fluid mechanics, Heat Transfer, Elasticity and Control theory may be offered as optional subjects. Undergraduate courses in Mathematics must be brought on par with the engineering courses.

Here are a few suggestions for the text book writers and teachers.

The topics should be presented in their historical perspective and the emphasis should be on understanding and not on rigour. The text book on differential equations written by George F. Simmons is an excellent example. Every definition and theorem should be preceded and followed by motivating examples and illustrations from the physical world or other areas of Mathematics. Wherever possible, biographical sketches can be provided. After all, mathematicians are also human beings!

Mathematics should not be projected as a bundle of theorems and proofs, but projected as a way of understanding the physical world.

An application oriented course is desirable from several points of view:

There are many problems in Physics, Engineering, Biology and Economics which require mathematical models for their solution. There are better employment opportunities from industries and research laboratories.

Applications provide excellent scope for further growth.

Finally, let us remember that it is the physical world that has enriched mathematics and mathematics lives only through applications.

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Stop the Tirade against Euclid Why Euclid should be more emphasized in Schools.

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This article purports to address the erroneous arguments in C.K. Raju's article: towards equity in Mathematics education 1. Goodbye Euclid! Published in Bharathiya Samajik Chintan, VII (4), new series and points out the need to emphasize Euclid more than it is done in classroom today. Euclid had been my best friend since High School (since I was 13). I enjoyed him by studying his theorems, especially problem solving. Solving a beautiful, challenging problem in geometry gives me a pleasure akin to listening a beautiful piece of music and infuses power in me to meet greater challenges.

In whatever stage I am at, in my mathematical pursuits, a good piece of result which is new to me, based on the Elements will propel me to finding more treasures hidden in it. With such a strong liking for Euclid, when someone attacks Euclid I cannot remain silent.

C.K.Raju who is a well-known computer scientist and physicist, in India and a friend of mine, has been arguing vehemently, to teach religiously neutral mathematics by banishing Euclid from present day school curriculum and eliminating formal mathematics from University Mathematics Curriculum first in

India and also world over!!!

This argument which has been around for a while, should be curbed right away looking at the ominous demands that Raju is making, (through his erroneous arguments, of course), before it corrupts young minds, further. I accidentally bumped into the article while browsing the net.

Now coming to the main issue, Raju's contention is the following: Theology influenced Greek Mathematics (especially the 'Elements'), by making it as he calls it 'Theologically Correct'. Since modern day formal mathematics is based on the Elements, Mathematics is essentially an instrument to perpetuate theological beliefs (more precisely, is only theological correct) and not universal and secular as is believed.

I call this total nonsense and absurd.

Making a mistake at a fundamental level, he builds a huge repertoire of arguments (in this paper and elsewhere) to arrive at totally false conclusions.

To bring back the poor Euclid, whom Raju has condemned to damnation (in his mind) let us start with Pythagoras.

It is believed by quite a few scholars starting from Voltaire, to present day US TV personalities like Albert Burke that Pythagoras in his extensive travels, also visited India.

Although Heroditus, (whom later historians call a Father of Lies) asserts that Pythagoras went to Egypt and learnt geometry from them, this view can be dismissed (although widely accepted), based on the fact that there is nothing in his symbolism that resembles to that in Egypt and some doubt

if he learnt anything from Egyptians at all.

I consider his visit to India a fact. His leanings towards Vedantic philosophy and Vegetarianism, strongly lend credence to this belief. He learned the theorem (which now bears his name Pythagoras Theorem) from the sulaba sutras on his visit to India and also learnt astronomy and vedantic philosophy by his contact with Vedic scholars in India and took his knowledge to Greece.

In other words, there was transmission of knowledge from India to Greece from his time. In fact, there was transmission of knowledge from India to other civilizations, like Egyptian and Babylonian in ancient times, but we dont need this observation for our arguments. It is also believed that a few other Greek scholars after him also visited India.

In other words, Vedantic wisdom and sciences like Mathematics and Astronomy found their way to Greece from India from the time of Pythagoras.

Greeks were great thinkers. They were the ones who initiated a scientific inquiry about the world around them through logical reasoning (or deductive proof) to satisfy the demands of logic and reasoning of the human mind.

These scholars like Pythagoras who brought the Vedantic wisdom of India to Greece believed on one hand they had [A] the knowledge of the Absolute truth (Vedic wisdom) and on the other hand [B] they were also interested in scientific pursuits, studying the world around them (Mathematics, Physics, Astronomy, etc.), in a way satisfying the demands of logic and reasoning. Hence, started the deductive process (or

deductive proof in Mathematics, starting with the Elements)

Some of them like Pythagoras, and Plato justified [B] on the basis that it would help in the realization of [A]. For other scholars [A] would not have mattered at all, like for e.g. in modern times like Newton, for whom theology mattered and Russell to whom theology was not of concern.

[A] and [B] are completely independent of each other. [A] had no influence on [B], although for some scholars [A] was a motivating factor for [B], after all the Elements consists of the combined results obtained by several mathematicians at that time. In any case, [A] and [B] are completely independent of each other although for some [A] was a motivating factor for [B].

In other words the pursuit of rational scientific inquiry, satisfying the demands of logic and reasoning (i.e. the method of deductive proof, from which arose the Elements) is completely independent of the theological beliefs of some Greek scholars and the subsequent theological meaning that they attached to its pursuit.

Surely for people like Pythagoras and Plato, and later Greeks, their pursuit of scientific knowledge through deductive reasoning (method of deductive proof) aided or supplemented their theological beliefs. Other than that [A] and [B] are not connected anyway.

To give a more mundane argument, if someone says that he or she wants to be the Prime Minister of India or president of US, in order to serve the people of their country, does the motive, have any influence on the entire process (technically,

the process) of getting elected? No. For some people it may be a motivating factor and for others it may just be a personal ambition.

Raju's fundamental mistake is this. Correct by reasoning (proof by deduction) and theological correctness are same (i.e. equivalent).

The fact is: Correct by reasoning (proof by deduction) lead to or implies theological correctness – [1]

Implication the other way is not true.

Theological correctness does not necessarily imply correct by reasoning - [2]

Speaking from the perspective of a lay person, as human beings, we clearly understand these facts. If something is reasonable or logical it doesn't conflict with theology (religious beliefs).

However, theological and religious beliefs are always not logical (do not always stand to reason). Thus, correct by reasoning and theological correctness are not equivalent.

Starting with Pythagoras and coming up to Einstein and later 20th Century physicists and mathematicians, Physics and Mathematics have lent a lot of credence to Vedic wisdom (theology). I quote an eminent American physicist Jon Archibald Wheeler, the first to involve in the theoretical development of the atomic bomb, in regards to 20th century physics he says "It is curious that people like Schroedinger, Niels Bohr and Oppenheimer are Upanishadic scholars"

This lends credence to [1] in the context of scientific inquiry in the sense that rigorous scientific inquiry starting with

the Elements (correct by reasoning) has finally led to the endorsement of Vedic wisdom (theological correctness).

The main thrust of Raju's arguments has been that logical correctness and theological correctness are equivalent (although this is not the way he states this, this is what is essentially implied by his arguments). To counter his arguments, let us go back 2300 years, and start with Euclid. As observed earlier, mathematics having been put on a reasonably sound footing starting with the Elements over centuries of development has aided physics of the 20th century in lending a lot of credibility to the truths expounded in the Vedas (Upanishads).

Assume you are a physicist, especially from India, (I am not a physicist, I am a Mathematician) and you are somehow uncomfortable with this outcome. One possible way out is accusing, the whole process of scientific enquiry having been orchestrated right at the beginning, so that the end results are not surprising, because at the very beginning the inquiry has been manipulated to produce the desired end result. This is precisely Raju's situation and his claims.

Unfortunately (for him) this is not the reality or situation. As I mentioned earlier, theology did nothing to influence the method of deductive proof (and hence the Elements) except for providing a motivation (and perhaps justification) for studying mathematics through deductive proof, for some of the early Greek mathematicians.

Let us visualize the whole situation by drawing a few diagrams. Let us call this a diagram describing the flow of scientific knowledge. It illustrates how starting with Euclid (Euclids

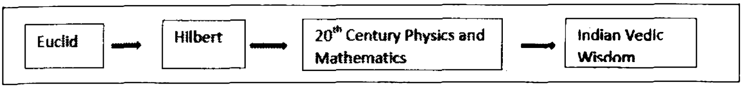

Figure 15

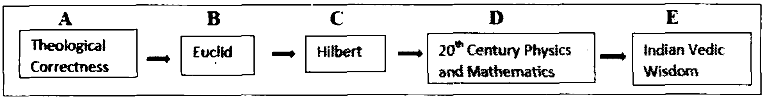

Figure 16

Elements) Mathematical knowledge progressed till the time of Hilbert when Euclids Elements were revised, to 20th Century physics and mathematics, finally lending credibility to Indian Vedic Wisdom. (I.e. logical correctness in science has finally led to theological correctness).

What Raju asserts is that, theological correctness, the result at the end, has been induced into Euclid (more precisely Euclids Elements) right at the beginning. Let us draw a diagram for this.

(Not a true state of affairs)
This diagram indicates that theological correctness has been induced into Euclid at the very beginning, but this is totally wrong. The reason is this, Euclid is a guy who accepts only logical arguments (starting with a set of axioms, which is the whole philosophy of the Elements). By saying that theological Correctness has been induced into Euclid, (more precisely into his Elements), of course implicitly, (because the domain of theology is different from the domain of the Elements, which is the physical world), what one is saying is that everything that

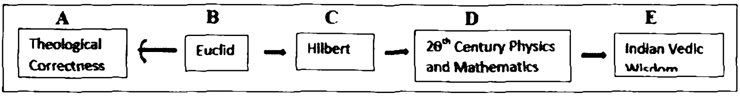

Figure 17

is theologically correct, is also logically correct, which is not true. Therefore, the arrow emanating from A to B does not exist. However, one can always draw an arrow from B to A, since anything that stands to logic does not contradict theology (which is definitely true). Let us draw a diagram for this

Now Raju is searching for a path from A to E to substantiate his fundamental (ulterior) emphasis, that the final outcome of centuries of scientific enquiry has been orchestrated right at the beginning. (I have spoken to him in 1980 when he was vehemently attacking vedic wisdom of India). However, no such path exists from A to E. Now comes the crucial step. How about we get rid of this guy Euclid? Since, details of Euclid are obscure, he can make use of that to establish that Euclid did not exist (and is just a fictitious name) and in such a case one can draw an arrow from A to B and sure then there is a path from A to E. That would establish his (wrong) claim that right at the beginning the final outcome has been manipulated by inducing theological correctness, so that the whole inquiry at the end lead (seems to confirm) to vedic wisdom (theological correctness). Whether he is doing this intentionally or unintentionally is besides the point. He is totally wrong, because Euclid is a genuine historical character who existed and wrote the Elements. The proof is easy. For this, we seek the help of Apollonius of Perga, the Great Geometer.

Apollonius wrote the conic sections a very original book about conics. The first four books in Greek have survived (please refer to Apollonius of perga history, Mac Tutor or T.L. Heath History of Greeks Mathematics), In book 3, he clearly mentions Euclid and his Elements. This is incontrovertible evidence about the existence of Euclid and his Elements and demolishes completely Raju's false claim that Euclid did not exist and is a fictitious name. Therefore it is clear that Euclid is a genuine historical character who wrote the Elements. The only doubt is his role in the authorship. He should have just been a compiler of the results in the Elements, occasionally providing missing details or proofs. Since it is a compilation of all available mathematical knowledge at that time, it could have been considered as an encyclopedia of Mathematics. Given this, it is very understandable why he is referred to as the Writer of the Elements by Proclus (which Raju cites in his paper), since Euclid cannot claim any ownership of the results except having compiled them. Since Euclid existed and wrote the Elements, therefore, in Fig 3, there is no arrow from A to B. This disproves Raju's claim that the Elements has been made theologically correct and hence, further, disproves the claim that formal mathematics itself has been made theologically correct. In fact it completely demolishes his entire thesis about the Cultural Foundations of Mathematics. There are a few more points (false claims) he makes elsewhere which I wish to address. 1) Raju says Mathematics is not universal and secular. Wrong. Mathematics is indeed universal and secular. To support his claim. When someone asked him if $2+2 = 4$, is not universal he counters it by saying that if you take 2 stones and 2 more stones and if you break 1 stone into 2 pieces then the above equation

does not hold. This is one of the most foolish arguments that I have ever heard. If you break one of the stones into 2, the left hand side of the equation is not $2+2$ anymore, it is $2+3$ which is 5. Further, in putting forth such an absurd argument he himself is using a 2-valued logic which he is opposing from the very beginning.

2) Now coming to his arguments, about 2-valued logic, our logic depends on our commonsense perception of the world. To give a crude example even a child knows that if he puts his hand in fire, he will get burnt, so he should not put his hand in fire. There is no such thing as he will, and at the same time he will not get burnt, the 3 or 4-valued logic he advocates. Coming to Mathematics, logic is of course 2-valued. Given the definition of an even and odd number, and given a number, it is either even or odd. Given a point and a line, either a point lies on it or it does not etc., isnt this obvious?

His argument is as follows. Either logic is decided 1) Culturally or 2) Empirically. If it is decided empirically, it depends on Physics. My question is why? I say this is false. It just depends on our commonsense perception of the world. Why should it depend on physics? Of course, no doubt, physics is the study of physical world.

If commonsense perception of the world does not agree with quantum physics, for which reason I think, he says quantum logic is not 2- valued, that is fine. This anomaly in 2-valued logic exists in Vedantic philosophy as well. If you approach a seer and tell him, based on Vedanta, since everything in this world is, Brahman, therefore I am God, the seer would probably tell you this is both true and false. One who has studied and

understood Vedantic philosophy can easily see the truth of this statement. Clearly, there is an anomaly with 2- valued logic here. Such anomalies exist at the philosophical level or in the case of science at the quantum level. If 2- valued logic decided empirically (based on common sense perception of the world) does not fit into quantum theory, that is fine. It doesn't have to. That does not in any way prove his conclusion that logic is decided culturally. (Further in putting forth such an argument, he is again using a 2-valued logic to substantiate an argument against 2-valued logic). This is totally nutty.

3. Raju mentions elsewhere that notions of infinity are tied to notions of eternity. Absurd again. The notion of infinity has nothing to do with time, and hence, eternity. One can convince a student of 8th grade about the notion of infinity by asking him to give the largest integer. If he comes up with one, you could ask him to add one more and get a larger integer. Where is time coming into the picture here? In fact, Euclid in his Elements, gives the proof of the infinitude of primes by assuming the existence of a largest prime and coming up with another prime bigger than that, proof, which an intelligent high school student can understand. Euclid existed before Christ and Christianity was born, to which Raju attributes the notion of eternity! 4) Elsewhere Raju also mentions the associative law of addition of numbers, in Mathematics, criticizing its acceptance since it involves an infinite process (adding infinite decimals). The foundations of real number system is made by the correspondence between real numbers and points on the line. Thus adding 3 real numbers is equivalent to adding 3 lengths which can be added in any order. This is the basis of the associative law. It does not

involve any infinite process. In fact the completeness of the real number system is established by such correspondence (For more on this please refer Hardys Pure Mathematics Chapter1). Also, Calculus does not depend on Physics (and hence on the notions of time and eternity). As is well known, It can be developed based on geometry. In fact, the very rudimentary traces of calculus the Method of Exhaustion developed by Archimedes, dates back to 3 centuries before Christ. Having established the existence of Euclid and his authorship of the Elements it is not necessary to dwell any further into Raju's erroneous arguments,But there is one more point I would like to make about the mystery geometry of Egypt to which he attributes the Elements. Egyptian Mathematics (including the mystery Mathematics) is computational in nature, sometimes even involving brute-force computations.There is not a trace of the type of deductive-proof Mathematics found in Egyptian Mathematics(including the mystery mathematics) as is carried out in the Elements.To attribute the Elements to Egyptians is indicative of a lack of proper understanding of ancient Egyptian Mathematics and in fact Mathematics as a whole. Now coming to the class room, both the deductive and empirical should be emphasized in the class room. I do not know the situation in India since I am in a faraway land. I feel the empirical and the computational aspects of Mathematics should be combined with deductive proof to give a more comprehensive view of Mathematics. Euclid should be more emphasized in schools than is currently done, since the goal is not just computation, but efficient computation. For this, more stress on deductive proof based mathematics (theorem proving) will provide students with the type of training needed for doing

efficient computation and devising new methods of efficient computation.

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A Study on Mathematics Curriculum and Achievement Level

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Introduction: The word *curriculum* is derived from Latin word currency which means to precede a race or course of a race. A curriculum may refer to the prescribed course of studies prepared to fulfill the aspiration of a society. So it cannot be a static one and has to be modified from time to time with the view of making education more relevant to the present day and future needs.

The Education was in the state list of Indian constitution up to 1976. Hence the respective State government was responsible for taking all decisions relating to school education including curriculum design, while Central Government is to help them in policy issues only. In 1976, the constitution was amended to include Education in concurrent list of Indian Constitution. As a result in the year 1986 National Policy on Education (NPE) was introduced. NPE 1986 introduced a common core curriculum for the country and National Council of Educational Research and Training (NCERT) was entrusted with the responsibility to prepare curriculum Frame Work for the country and to review from time to time as per needs and demands of the time.

National Council of Educational Research and Training decided to revise the National Curriculum Frame work for School Education-2000 (NCFSE-2000) in the light of the

report Learning without Burden and formulated the National Curriculum Framework 2005 (NCF-2005). In the NCF-05 significant changes of learning Mathematics were recommended with a view of making education more relevant to the present day and future needs. NCF-2005 proposes five principles for curriculum development: i) Connecting knowledge to life outside the school. ii) Ensuring that learning shifts away from rote method iii) Enriching the curriculum so that it goes beyond textbooks. iv) Making examination more flexible and integrating them with classroom life and v) Nurturing an overriding identity informed by caring concern within the democratic policy of the country.

NCF 2005 emphasized on shifting focus of mathematics learning from achieving 'narrow' to 'higher' goal. The 'narrow' aim of Mathematics Learning is to develop useful capabilities, more specifically those relating to numeracy – number, operations of numbers, measurement, decimal , percentage etc. The higher aim of Mathematics is the Mathematisation of child. It refers to ability to think logically, formulation and abstraction.

This paper aims to study the impact of the implementation of the above changes in the curriculum. More precisely, we have the following objective:

- To know the achievement level of secondary school learners in terms of Mathematical goal specified in National Curriculum Frame Work 2005.

We made the following **Hypothesis:**

1. The Achievement level of major section of Learners even those scored higher marks in at Class 10 Public Examination in Mathematics are not at par to continue higher study in Mathematics.

2. The achievement level of curricular objectives among the learners in mathematics may be improved by changing class room transaction through modification of mathematics textbook.

Methodology: The study is confined to the study of learners of secondary schools who scored good marks (above 80%) in Mathematics at school based evaluation. Thirty students of Class VIII, IX and X were selected from various schools with representation of students from Vernacular to English medium, rural area to urban area and private to government school learners. The learning outcome was evaluated in certain learning points which are desirable for the learners of the level or below the level. Finally a sharing with them was conducted to know about their classroom learning and support from their parents.

Experimental/Evaluation Parameters: As a part of study an evaluation was conducted among the sample learners on learning points comprising from both narrow and higher goal of Mathematics learning. Learning points evaluated from the area of narrow Mathematical Goal:

1. Basic Mathematical operations in rational numbers, fraction and decimals.

2. Plotting of fractions and decimal numbers in number Line.

3. Square root and cube root of real numbers.
4. Division of algebraic expression by another algebraic expression.

Areas covered to evaluate the achievement level of learners in terms of Higher Curricular goal of Mathematics.

1. Make similarity and dissimilarity among the different quadrilaterals with the reasons.
2. If angle bisector of an angle of a triangle is also a median of the triangle then find the altitude.
3. Some Geometrical proofs relating to different types of quadrilaterals and circle.
4. Formation of equation with single and two variables for some selected types of word problems.

Findings: Although the students scored good marks in their school based evaluation, the average achievement level even in narrow curricular goal of some basic learning points are not achieved on an average. At the same time achievement level of learners in the higher Mathematical objectives are poor among learners. It is really a very disappointing state that such a performance is seen even among the so called brilliant students. The major points emerging from the sharing with the students are given in the tables below.

Results

Table 1:

Evaluation data for achievement of narrow Mathematical Goal

Learning Points	Achieved Learning Points		Non-achieved Learning Points	
	In Nos	In (%)	In Nos	In (%)
Basic Mathematical operations in rational numbers, fractions and decimals.	22	73.33%	8	26.67%
Plotting of fractions and decimal numbers in Number line.	17	56.67%	13	43.33%
Square root and cube root of real numbers	20	66.67%	10	33.33%
Division of algebraic expression by another algebraic expression.	13	43.33%	17	56.67%

Table 2:
Evaluation data for evaluation of higher Mathematical Goal

Learning Points	Achieved Learning Points		Non-achieved Learning Points	
	In Nos	In (%)	In Nos	In (%)
Make similarity and dissimilarity among the different quadrilaterals with the reasons.	12	40%	18	60%
If angle bisector of an angle of a triangle is also a median of the triangle then find the altitude. For what types of triangles it is satisfied and why?	8	26.67%	22	73.33%
Geometrical proof relating to different types of quadrilaterals and circles.	7	23.33%	23	76.67%
Formation of equations single and two variables for some selected types of word problems.	9	30%	21	70%

The following observations can be made:

1. They are taught at school the process to solve problems given in the textbook and they are asked to solve same kind of problems using the process.

2. They are not able to solve problem which are not taught in classroom.

3. They practice the solved problems repeatedly to remember it for examination.

4. Every student of the sample group goes for private tuition in Mathematics and prepare for examination by selecting questions from question banks.

Conclusion: It is known to all that until and unless the higher mathematical objectives are not addressed properly in the schooling system we cannot even think for achievement of narrow Mathematical goal. The National Curriculum Framework-2005 has also given emphasis on this point. Unfortunately, still most of our school students are taught about the steps and process of solving mathematical problems given in the text book. Accordingly students memorize the same and recall it at the Examination to score well. The so called performance influences the parents a lot to encourage their child to follow this practice. They appreciate the teacher along with the child for the performance and at the same time advise others to follow the teacher. In this way a teacher is recognized as a good teacher in the society. On the contrary the teachers who concentrate on the achievement of the actual curricular goal have to face a lot of problems from every part even from the authorities. The teachers become frustrated and are compelled to adopt the high scoring teaching procedure. Further the perception among the common people that Mathematics is a very difficult subject and needs special attention leads to provision of excessive support on the subject

through the coaching classes or private tuitions. The way of repeated practice, hard work in preparation with selected questions and excessive support help them to score good marks in the subject in class 10 public examinations. It definitely reduces the validity and reliability parameter of our public examination. From the above discussion although it is not conclusive to generalize for all but still most of our schools and students adopt this process. Moreover, the parents and social environment motivate the students who brilliantly performed in mathematics at class 10 public examination to go for science stream and lot many times they get encouraged to select the stream without their interest and capability, for which they failed to achieve desirable outcome in their study resulting in frustration among them and ultimately the country suffers from lack of proper human resources in different fields.

As our education system is too much dependent on textbooks rather than learning outcomes, it may be expected that by developing textbooks having provision of learning with activities indicating the purpose and guidelines for execution of the activities will definitely help to change the classroom transaction to a certain extent.

A sample activity may be as follows:

Class: II

Learning points: Subtraction; Sub learning Points: Comparison of two numbers

Activity: Teacher will give 7 marbles to one student and 4 marbles to another student and then ask the first student how many more marbles has he got than the other student. Teacher will continue same type of activity with different number of

marbles.
After few days teacher will switch them to written problem like "By how much 7 is greater than 3?" use larger numbers.

It is to be mentioned here that the second hypothesis could not be tested for want of time.

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On Mamikon's Theorem

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1. Introduction

Let Γ be a smooth bounded curve in the plane. The tangent segments of Γ that are of constant length defines a region that is bounded by a curve called the tangent curve τ (see Figure 18). Note that such a tangent curve arises for example, as the

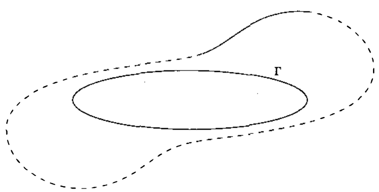

Figure 18: Elliptical Inner Curve

curve traced by the front wheel of a bicycle when the rear wheel traces the curve Γ.

In [6] Mamikon proved a very interesting "Sweeping tangent theorem" that computes the area between Γ and τ in an innovative way. The region between Γ and τ is called tangent sweep and Mamikon defines tangent clusters corresponding to a tangent sweep by translating each tangent to Γ parallel to itself to bring the points of tangency to a fixed point. He proves the following interesting theorem:

Mamikon's Theorem The area of a tangent sweep is equal

to the area of its tangent cluster regardless of the shape of the original curve.

The theorem is valid for a space curve also and the proof involves tools from advanced differential geometry. This theorem is interesting in its own right even for plane curves and in order that it can be appreciated even by those who have only an elementary knowledge of differential geometry, this paper aims to give a proof of the above theorem when Γ is a plane curve and when tangents are of constant length.

In Section 2 below we prove the theorem for the special case of elliptical inner curve using Green's theorem. In Section 3, we give proof of Mamikon's theorem for constant length tangent sweeps.

2. A special case – Ellipse as inner curve

In order to understand the complexity of the problem, let us first assume that the inner curve Γ is an ellipse. We will compute the are of the tangent sweep using parametric equations.

Let the equation of the inner ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. If $\mathbf{r}$ is the position vector of any point P on the ellipse, we can write $\mathbf{r} = a \cos t\mathbf{i} + b \sin t\mathbf{j}$. The tangent vector at P is given by $\frac{d\mathbf{r}}{dt} = -a \sin t\mathbf{i} + b \cos t\mathbf{j}$. Hence the unit vector $\hat{\mathbf{t}}$ in the direction of the tangent can be written as

$$\hat{\mathbf{t}} = \frac{-a \sin t\mathbf{i} + b \cos t\mathbf{j}}{\sqrt{a^2 \sin^2 t + b^2 \cos^2 t}}$$

The position vector $X\mathbf{i} + Y\mathbf{j}$ of the point at which the tangent

to the ellipse meets the outer curve is given by

$$X\mathbf{i} + Y\mathbf{j} = \mathbf{r} + \ell\hat{\mathbf{t}}$$

where ℓ is the length of the wheelbase of the bicycle. Thus we have

$$X = a\left(\cos t - \frac{\ell \sin t}{\sqrt{a^2 \sin^2 t + b^2 \cos^2 t}}\right),$$
$$Y = b\left(\sin t + \frac{\ell \cos t}{\sqrt{a^2 \sin^2 t + b^2 \cos^2 t}}\right)$$

To simplify the computation, we will use Green's Theorem that states that area bounded by a closed curve Γ is given by the line integral

$$\frac{1}{2}\int_\Gamma X dY - Y dX$$

We have

$$\frac{dX}{dt} = a\left\{-\sin t - \ell\left(\frac{\cos t}{\sqrt{a^2 \sin^2 t + b^2 \cos^2 t}}\right) + \ell \sin t\left(\frac{(a^2 - b^2)\sin t \cos t}{(a^2 \sin^2 t + b^2 \cos^2 t)^{\frac{3}{2}}}\right)\right\}$$

$$\frac{dY}{dt} = b\left\{\cos t - \ell\left(\frac{\sin t}{\sqrt{a^2 \sin^2 t + b^2 \cos^2 t}}\right) - \ell \cos t\left(\frac{(a^2 - b^2)\sin t \cos t}{(a^2 \sin^2 t + b^2 \cos^2 t)^{\frac{3}{2}}}\right)\right\}$$

Area bounded by the outer curve is given by

$$
\begin{aligned}
\frac{1}{2}\int & XdY - YdX \\
&= \frac{ab}{2}\int_0^{2\pi} (\sin^2 t + \cos^2 t)\, dt \\
&\quad - \frac{ab}{2}\int_0^{2\pi} \frac{\ell(a^2 - b^2)(\sin t \cos^3 t + \cos t \sin^3 t)}{(a^2 \sin^2 t + b^2 \cos^2 t)^2}\, dt \\
&\quad + \frac{ab}{2}\int_0^{2\pi} \frac{\ell^2(\sin^2 t + \cos^2 t)}{a^2 \sin^2 t + b^2 \cos^2 t}\, dt \\
&= \pi ab - \frac{ab}{2}\int_0^{2\pi} \frac{\ell(a^2 - b^2)\sin t \cos t}{(a^2 \sin^2 t + b^2 \cos^2 t)^2}\, dt \\
&\quad + \frac{ab}{2}\int_0^{2\pi} \frac{\ell^2}{a^2 \sin^2 t + b^2 \cos^2 t}\, dt
\end{aligned}
$$

The integral in the middle is easily seen to be zero and the last term evaluates to $\pi\ell^2$. Thus the area bounded by the outer curve is $\pi ab + \pi\ell^2$. Since the area of the ellipse is πab, the area between the tracks is $\pi\ell^2$.

The fact is that the area can be found using a clever and beautiful "visual technique" with no computation – yes – without using integration. This technique is due to Mamikon A.Mnatsakanian [6]. Also, the quantity $\pi\ell^2$ is the area of a circle with radius ℓ. We will see that the area of the tangent sweep of constant length tangents for any closed simple curve is indeed equal to the area of the circle with the radius equal to the length of the tangent.

3. Mamikon's Theorem – Special case of constant length tangent

Let Γ be a smooth curve in the plane. In this section, we prove that the area swept by a tangent of constant length to Γ is

equal to the area of the circular sector obtained by translating the tangents parallel to themselves so that they all have the same initial point (Figure 19).

In [6], an intuitive proof using polygon approximations is given.

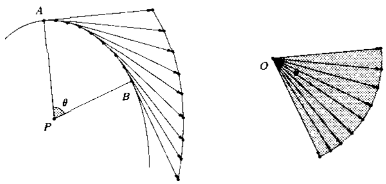

Figure 19: Tangent Sweep

We give a proof using elementary differential geometry.

We can parametrize Γ using the arc length s. If $\mathbf{x}(s)$ is the position vector of any point on Γ, then the tangent vector $\mathbf{t}(s)$ is of unit length and we have $\mathbf{t}'(s) = \kappa(s)$ where κ is the curvature of Γ.

Lemma([8]) Let Γ be any smooth curve and let $\Delta\theta$ be the angle between the unit tangent $\mathbf{t}(s)$ at $\mathbf{x}(s)$ and $\mathbf{t}(s+\Delta s)$ at a neighboring point $\mathbf{x}(s+\Delta s)$, $\Delta s > 0$. Then the curvature

$$|\kappa| = \lim_{\Delta s \to 0} \frac{\Delta\theta}{\Delta s} = \frac{d\theta}{ds}$$

Proof Since $\mathbf{t}$ is a unit vector, $|\mathbf{t}(s+\Delta s) - \mathbf{t}(s)|$ is the base of an isosceles triangle with sides of length 1 (as shown in Figure 20). Hence

$$|\mathbf{t}(s+\Delta s) - \mathbf{t}(s)| = 2\sin\left(\frac{\Delta\theta}{2}\right) = \Delta\theta + o(\Delta\theta)$$

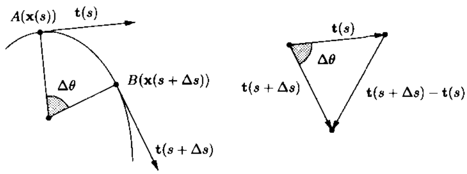

Figure 20

using the Taylor expansion for the sine function.

$$|\kappa| = \lim_{\Delta s \to 0} \left| \frac{\mathbf{t}(s+\Delta s) - \mathbf{t}(s)}{\Delta s} \right| = \lim_{\Delta s \to 0} \left| \frac{\Delta\theta + o(\Delta\theta)}{\Delta s} \right| = \frac{d\theta}{ds}$$

Without losing generality, we can assume that the sweeping tangent is of unit length. From the lemma above, the area of the triangle formed by the translated tangents at two nearby points is $\frac{1}{2}|\mathbf{k}(\mathbf{s})|\Delta s$.

Consider the tangents at nearby points $P(\mathbf{x}(s))$ and $P'(\mathbf{x}(s+\Delta s))$ and the area of the triangle PQR in Figure 21, which is nearly equal to the area swept by the tangent. We will show that this area is nearly equal to $\frac{1}{2}|\mathbf{k}(\mathbf{s})|\Delta s$, and going to the limit as $\Delta s \to 0$, we obtain the desired result.

We have

$$\begin{aligned} \vec{PR} &= \mathbf{t}(s+\Delta s) + \mathbf{x}(s+\Delta s) - \mathbf{x}(s) \\ &\approx \kappa(s)\Delta s + \mathbf{t}(s) + \mathbf{t}(s)\Delta s \end{aligned}$$

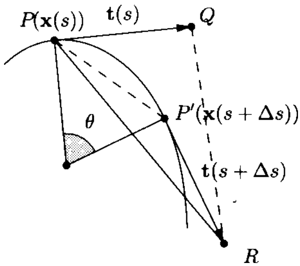

Figure 21

Hence

$$|\vec{PR} \times \vec{PQ}| \approx |\kappa(s) \times \mathbf{t}(s)|\Delta s = |\kappa(s)|\Delta s,$$

since $\kappa(s)$ is perpendicular to $\mathbf{t}(s)$. Thus the area of the triangle $PQR \approx \frac{1}{2}|\kappa(s)|\Delta s$. Note that the approximation involved is $o((\Delta s)^2)$ and hence in the limit, we obtain that the area swept by the tangents is the same as the area of the circular sector formed by the translated tangents.

From this it readily follows that when the rear wheel of the bicycle follows any oval curve, the area between the tracks is equal to the area of a circle with radius equal to the wheel base of the bicycle.

In fact, Mamikon's theorem is true for space curves as well and also for varying tangent lengths (see [7] for a beautifully illustrated discussion on this).

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A Simpler Proof of Polya's Theorm on Symmetric Random Walks

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Polya's theorem on symmetric random walks is stated as follows:

Theorem 1 *A symmetric random walk is recurrent in dimensions 1 and 2 but transient in dimensions* ≥ 3.

In the proof of the above theorem, one uses Stirling's approximation for $n!$ ([1], [2], [3], [4]) . Students often find Stirling's approximation non-intuitive. In this note, we provide a more elementary approach to Polya's theorem.

Consider a completely drunk person who walks along a street. Being drunk, he has no sense of direction. So he may move forwards with equal probability that he moves backwards. This is symmetric random walk in one dimension. We can also consider random walk in higher dimensions. We can assume a grid of streets and the drunk can move one step in any of the four directions with equal probability. In three dimensions all the six possible directions are taken with equal probability of $\frac{1}{6}$.

A random walk (in any dimension) is said to be recurrent if the probability of escape – that is, probability that the walk never returns to 0 is zero and transient if the probability of escape is positive. If w_n is the probability that the walk returns to the

starting point in n steps, we have (see [2])

$$\text{Recurrence} \iff \sum_0^\infty w_n \text{ diverges}$$

$$\text{Transience} \iff \sum_0^\infty w_n \text{ converges}$$

Let $w_{i,n}$ denote the probability that in dimension i that the walk returns to 0 at the n^{th} step. Clearly $w_{i,2n+1} = 0$. It is easy (See [2]) to show that the probabilities $w_{i,2n}$ for dimension $i = 1, 2, 3$ are given respectively by

$$\begin{aligned}
w_{1,2n} &= \binom{2n}{n}\left(\frac{1}{2}\right)^{2n} \\
w_{2,2n} &= \left(\binom{2n}{n}\left(\frac{1}{2^{2n}}\right)\right)^2 \\
w_{3,2n} &= \left(\frac{1}{2}\right)^{2n}\binom{2n}{n}\sum_{i=0,j=0,i+j\le n}^{n,n}\left(\frac{1}{3^n}\frac{n!}{i!j!(n-i-j)!}\right)^2
\end{aligned}$$

For dimension 1 we have,

$$\begin{aligned}
w_{1,2n} &= \frac{1}{2^{2n}}\binom{2n}{n} \\
&= \frac{1}{2^{2n}}\frac{1\cdot 2\cdots(2n-1)\cdot(2n)}{(1\cdot 2\cdots n)(1\cdot 2\cdots n)} \\
&= \frac{1}{2^{2n}}\frac{(1\cdot 3\cdot 5\cdots(2n-1))(2\cdot 4\cdots(2n))}{(1\cdot 2\cdots n)(1\cdot 2\cdots n)} \\
&= \frac{1}{2}\cdot\frac{3}{4}\cdots\frac{2n-1}{2n}
\end{aligned}$$

Hence

$$\begin{aligned}
w_{1,2n}^2 &= \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{5}{6}\cdots\frac{2n-1}{2n}\cdot\frac{2n-1}{2n} \\
&\ge \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{4}{5}\cdot\frac{5}{6}\cdots\frac{2n-2}{2n-1}\cdot\frac{2n-1}{2n} \\
&= \frac{1}{4n}
\end{aligned}$$

Hence $w_{1,2n} \geq \frac{1}{2\sqrt{n}}$ and $\sum \frac{1}{\sqrt{n}}$ diverges, the walk in one dimension is recurrent.

For dimension 2, we have

$$w_{2,2n} = \left(\binom{2n}{n}\left(\frac{1}{2^{2n}}\right)\right)^2 \geq \frac{1}{4n}$$

and again $\sum w_{2,2n}$ diverges and hence the walk in two dimensions is also recurrent.

For dimension 3, we have

$$w_{3,2n} = \left(\frac{1}{2}\right)^{2n}\binom{2n}{n}\sum_{i=0,j=0,i+j\leq n}^{n,n}\left(\frac{1}{3^n}\frac{n!}{i!j!(n-i-j)!}\right)^2$$

Let M be the maximum of

$$\left(\frac{1}{3^n}\frac{n!}{i!j!(n-i-j)!}\right)$$

Note that

$$\sum_{i=0,j=0,i+j\leq n}^{n,n}\frac{n!}{i!j!(n-i-j)!}$$

is the number of ways of placing n numbered balls in 3 numbered urns and hence equals 3^n. Thus

$$w_{3,2n} \leq M\left(\frac{1}{2}\right)^{2n}\binom{2n}{n}$$

Now clearly, M is obtained when i, j and $n-i-j$ are close to $\frac{n}{3}$. Thus

$$w_{3,2n} \leq \left(\frac{1}{2}\right)^{2n}\binom{2n}{n}\left(\frac{1}{3^n}\frac{n!}{[\frac{n}{3}]![\frac{n}{3}]![\frac{n}{3}]!}\right)$$

Now,

$$\begin{aligned}\left(\frac{1}{2^{2n}}\binom{2n}{n}\right)^2 &= \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{5}{6}\cdots\frac{2n-1}{2n}\cdot\frac{2n-1}{2n}\\ &\leq \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{4}{5}\cdot\frac{5}{6}\cdot\frac{6}{7}\cdots\frac{2n-1}{2n}\cdot\frac{2n}{2n+1}\\ &= \frac{1}{2n+1} < \frac{1}{2n}\end{aligned}$$

Also, letting $n = 3k$, we have

$$
\begin{aligned}
\left(\frac{1}{3^n}\frac{n!}{[\frac{n}{3}]![\frac{n}{3}]![\frac{n}{3}]!}\right) &= \frac{1}{3^{3k}}\frac{(3k)!}{k!k!k!} \\
&= \frac{1}{3}\cdot\frac{2}{3}\cdot\frac{3}{3}\cdots\frac{3k-2}{3k}\cdot\frac{3k-1}{3k}\cdot\frac{3k}{3k} \\
&\leq \frac{1}{3}\cdot\frac{3}{4}\cdot\frac{4}{4}\cdots\frac{3k-2}{3k}\cdot\frac{3k}{3k+1}\cdot\frac{3k+1}{3k+1} \\
&= \frac{1}{3k+1} < \frac{1}{n}
\end{aligned}
$$

Hence we obtain

$$
w_{3,2n} \leq \left(\frac{1}{2}\right)^{2n}\binom{2n}{n}\left(\frac{1}{3^n}\frac{n!}{[\frac{n}{3}]![\frac{n}{3}]![\frac{n}{3}]!}\right) \leq \frac{1}{\sqrt{2n}}\cdot\frac{1}{n}
$$

and hence

$$
\sum w_{3,2n} \leq \sum \frac{1}{\sqrt{2}n^{\frac{3}{2}}}
$$

and converges. Thus the walk in three dimensions is transient.

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An Intuitive Approach to Product of Gauss Functions

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Consider the integral

$$I = \int_{-\infty}^{+\infty} e^{-x^2} dx \tag{1}$$

For the three Cartesian directions x, y, z of a right handed coordinate system, we have the product

$$I^3 = \int_{x=-\infty}^{+\infty} \int_{y=-\infty}^{+\infty} \int_{z=-\infty}^{+\infty} e^{-(x^2+y^2+z^2)}(dxdydz),$$

Changing to spherical polar coordinates, we have

$$I^3 = \int_{r=0}^{\infty} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} e^{-r^2}(r^2 \sin\theta) f(\phi)(drd\theta d\phi) \tag{2}$$

$$= \int_{r=0}^{\infty} e^{-r^2}(4\pi r^2)dr = 4\pi \int_{r=0}^{\infty} e^{-r^2} r^2 dr \tag{3}$$

From (1) given above,

$$I = 2\int_0^{\infty} e^{-x^2} dx$$

which, by integration by parts, gives

$$I = 2\left\{2\left((xe^{-x^2})_0^{\infty} + 2\int_0^{\infty} x^2 e^{-x^2} dx\right)\right]$$

The first term within the bracket on the right hand side vanishes at both limits. Therefore,

$$\frac{I}{4} = \int_0^{\infty} e^{-x^2} x^2 dx \tag{4}$$

Hence from (3) and (4), we have

$$\frac{I^3}{4\pi} = \frac{I}{4}$$

Thus we get the well-known result:

$$I = \int_0^{\infty} e^{-x^2} dx = \sqrt{\pi} \tag{5}$$

Now, from

$$I^2 = 2\pi \int_0^{\infty} e^{-r^2} r dr \tag{6}$$

$$I^3 = 4\pi \int_0^{\infty} e^{-r^2} r^2 dr \tag{7}$$

we can conjecture intuitively that

$$I^{2s} = (\Omega) \int_0^{\infty} e^{-r^2} r^{2s-1} dr \tag{8}$$

Where "Ω" is a coefficient obtained by $(2s - 1)$ integrations of relevant variables, each over its limits.

At this stage our approach is "purely intuitive". (It is not regarded in any sense, as a rigorous proof). However equation (5) helps us to move further. Putting $r^2 = z$, we get

$$\begin{aligned} I^{2s} &= (\Omega) \int_0^{\infty} e^{-r^2} r^{2s-1} dr \\ &= \frac{1}{2} \int_0^{\infty} e^{-z} z^{s-1} dz \\ &= \frac{1}{2} \Gamma(s) \end{aligned}$$

using the standard Gamma function. Also, intuitively, $I^{2s} = \pi^s$. Thus we can write

$$\frac{\Omega \times \Gamma(s)}{2} = \pi^s$$

or

$$I^{2s} = \frac{2\pi^s}{\Gamma(s)} \int_0^\infty e^{-r^2} r^{2s-1} dr \tag{9}$$

Using the properties of the Gamma function, namely,

$$\Gamma(n+1) = n\Gamma(n),\ \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

we easily verify that (9) holds for small values of s.

When $2s = 1$, we have

$$I = \frac{2\sqrt{\pi}}{\Gamma\left(\frac{1}{2}\right)} \int_0^\infty e^{-r^2} dr = 2\int_0^\infty e^{-r^2} dr$$

since $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.
When $2s = 2$, we have

$$I^2 = \frac{2\pi}{\Gamma(1)} \int_0^\infty e^{-r^2} r dr = 2\pi \int_0^\infty e^{-r^2} r dr$$

When $2s = 3$, we have

$$I^3 = \frac{2\pi^{\frac{3}{2}}}{\Gamma\left(\frac{3}{2}\right)} \int_0^\infty e^{-r^2} r^2 dr = 4\pi \int_0^\infty e^{-r^2} r^2 dr$$

Since

$$I = 2\int_0^\infty e^{-r^2} dr$$

by repeated integration by parts, we get

$$\begin{aligned} I &= \frac{2}{\frac{1}{2} \times \frac{3}{2}} \int_0^\infty e^{-r^2} r^4 dr \\ &= \frac{2}{\frac{1}{2} \times \frac{3}{2} \times \frac{5}{2}} \int_0^\infty e^{-r^2} r^6 dr \\ &= \cdots \end{aligned}$$

Hence we can intuitively obtain

$$I^{2s+1} = (\Omega)\int_0^\infty e^{-r^2} r^{2s} dr$$

Integrating by parts repeatedly and rearranging, we get

$$\left(\frac{I}{2}\right) \times \left(\frac{1}{2} \times \frac{3}{2} \cdots \times \frac{2s-1}{2}\right) = \int_0^\infty e^{-r^2} r^{2s} dr$$
$$= \frac{I^{2s+1}}{\Omega}$$

Hence

$$\frac{I^{2s}}{\Omega} = \frac{1}{2\sqrt{\pi}}\Gamma\left(\frac{2s+1}{2}\right) = \frac{\pi^{2s}}{\Omega}$$

Hence

$$\Omega = 2\frac{\pi^{s+\frac{1}{2}}}{\Gamma\left(s+\frac{1}{2}\right)}$$

Also, from

$$I^{2s} = \frac{2\pi^s}{\Gamma(s)}\int_0^\infty e^{-r^2} r^{2s-1} dr$$

we see that the volume element in $2s$ dimensional space can be written as

$$dV = \frac{2\pi^s}{\Gamma(s)} r^{2s-1} dr$$

Let us check this in small dimensions:

In two dimensions, we have $s = 1$ and

$$dV = 2\pi r dr$$

In three dimensions, $s = \frac{3}{2}$ and

$$dV = \frac{2\pi^{\frac{3}{2}}}{\Gamma\left(\frac{3}{2}\right)} r^2 dr = 4\pi r^2 dr$$

Similarly, we have

$$s = 2,\ dV = 2\pi^2 r^3 dr$$

in four dimensions.

These show that the intuitive approach also yields the same results as obtained by conventional approach.

Sum of r^{th} powers of an Arithmetic Progression

Faiz Imam

Let

$$a, a+d, a+2d, \ldots$$

be an arithmetic progression with common difference $d \neq 0$. Let r be a natural number. We compute

$$\Delta = a^r + (a+d)^r + (a+2d)^r + \cdots + (a+(n-1)d)^r$$

the sum of $r-$ th powers of the first n terms of the progression. Let $\alpha = \frac{a}{d}$. The above sum can be written as

$$\Delta = d^r\left(\alpha^r + (1+\alpha)^r + (2+\alpha)^r + \cdots + (n-1+\alpha)^r\right)$$

Put

$$S = e^{\alpha x} + e^{(1+\alpha)x} + e^{(2+\alpha)x} + \cdots + e^{(n-1+\alpha)x} = \sum_{k=0}^{n-1} e^{(k+\alpha)x}$$

Then

$$\frac{d^r S}{dx^r} = \sum_{k=0}^{n-1}(k+\alpha)^r e^{(k+\alpha)x}$$

and

$$\left.\frac{d^r S}{dx^r}\right|_{x=0} = \sum_{k=0}^{n-1}(k+\alpha)^r$$

Now,

$$\begin{aligned}
S &= e^{\alpha x} + e^{(1+\alpha)x} + e^{(2+\alpha)x+\cdots+e^{(n-1+\alpha)x}} \\
&= e^{\alpha x}(1 + e^x + e^{2x} + \cdots + e^{(n-1)x}) \\
&= e^{\alpha x}(1 + t + t^2 + \cdots + t^{n-1}), \quad \text{where } t = e^x \\
&= t^\alpha \frac{t^n - 1}{t - 1}
\end{aligned}$$

Hence

$$(t-1)S = t^{n+\alpha} - t^{\alpha}$$

For any positive integer p, we have

$$\begin{aligned}\frac{d^{p+1}}{dt^{p+1}}((t-1)S) &= (t-1)\frac{d^{p+1}S}{dt^{p+1}} + \binom{p+1}{1}\frac{d^pS}{dt^p} \\ &= (n+\alpha)(n+\alpha-1)\cdots(n+\alpha-p)t^{n+\alpha-p-1} \\ &\quad - \alpha(\alpha-1)\cdots(\alpha-p)t^{\alpha-p-1}\end{aligned}$$

Hence at $t = 1$, we have

$$\left.\frac{d^pS}{dt^p}\right|_{t=1} = \frac{(n+\alpha)(n+\alpha-1)\cdots(n+\alpha-p) - \alpha(\alpha-1)\cdots(\alpha-p)}{p+1}$$

Since $t = e^x$, $\dfrac{dt}{dx} = e^x = t$. Thus

$$\frac{dS}{dx} = \frac{dS}{dt}\frac{dt}{dx} = t\frac{dS}{dt}$$

Hence

$$\begin{aligned}\frac{d^2S}{dx^2} &= \frac{d}{dt}\left(t\frac{dS}{dt}\right)\frac{dt}{dx} \\ &= t^2\frac{d^2S}{dt^2} + t\frac{dS}{dt}\end{aligned}$$

Suppose that we have found constants C_k^p such that

$$\begin{aligned}\frac{d^pS}{dx^p} &= C_1^p t\frac{dS}{dt} + C_2^p t^2\frac{d^2S}{dt^2} + \cdots + C_p^p t^p\frac{d^pS}{dt^p} \\ &= \sum_{k=1}^{p} C_k^p t^k \frac{d^kS}{dt^k}\end{aligned}$$

then

$$\begin{aligned}\frac{d^{p+1}S}{dx^{p+1}} &= t\frac{d}{dt}\left(\sum_{k=1}^{p} C_k^p t^k\frac{d^kS}{dt^k}\right) \\ &= t\sum_{k=1}^{p} C_k^p\left(kt^{k-1}\frac{d^kS}{dt^k} + t^k\frac{d^{k+1}S}{dt^{k+1}}\right)\end{aligned}$$

Hence we have

$$C_k^{p+1} = kC_k^p + C_{k-1}^p$$

The numbers C_k^p are called the Stirling numbers of the second kind. These are readily computed:

$$C_1^1 = 1,\ C_1^2 = 1, C_2^2 = 1$$
$$C_1^3 = 1,\ C_2^3 = 3, C_3^3 = 1$$

and so on. Now,

$$\begin{aligned}\sum_{k=1}^{n-1}(\alpha + k)^r &= \left.\frac{d^r S}{dx^r}\right|_{x=0} \\ &= \sum_{k=1}^{r} C_k^r t^k \left.\frac{d^k S}{dt^k}\right|_{t=1} \\ &= \sum_{k=1}^{r} C_k^r \frac{(n+\alpha)\cdots(n+\alpha-k) - \alpha\cdots(\alpha-k)}{k+1}\end{aligned}$$

Since $\alpha = \dfrac{a}{d}$, we have

$$\sum_{k=1}^{n-1}(a+kd)^r$$
$$= d^r \sum_{k=1}^{r} C_k^r \frac{(a+nd)\cdots(a+(n-k)d) - a\cdots(a-kd)}{(k+1)d^{k+1}} \tag{1}$$

Thus (1) gives the sum of the first n terms of an arithmetic progression. Using (1), we can deduce the following well-known results:

Corollary

$$1^r + 2^r + \cdots + n^r = \sum_{k=1}^{r} C_k^r \frac{(n+1)n\cdots(n-k+1)}{k+1}$$

Corollary

$$\begin{aligned}1^2+3^2+\cdots+(2n-1)^2 &= 2^2\left(C_1^2\frac{(2n+1)(2n-1)+1}{2\cdot 2^2}\right.\\ &\quad\left.+C_2^2\frac{(2n+1)(2n-1)(2n-3)-1\cdot 3}{3\cdot 2^3}\right)\\ &= \frac{1}{3}n(2n+1)(2n-1)\end{aligned}$$

I wish to thank prof K.C. Prasad who helped in preparation of this paper.

The following papers also consider sum of powers of terms of an arithmetic progression.

1. Frank Chorlton, Finite sums of powers of the natural numbers, Math Gaz. 83, pp 95-96.

2. Martin Griffiths Sums of powers of the terms in any finite Arithmetic Progression, Math Gaz 86, pp 269-271

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ASSOCIATION ACTICVITES

1. The **49th conference of** the AMTI took place **at Saraswathi Vidya Mandir Hr. Sec. Residential School, Sharda Vihar, Bhopal** on **27th , 28th and 29th Dec. 2014** . 154 outstation and **42 local participants** participated as delegates.

   Sri P.S. Kalra President, Sahodaya school complex, Bhopal opened the exhibition, released the souvenir and inaugurated the conference after prayer and lighting the lamp.

   Prof. Rajendra Bhatia, President, Prof. J. Pandurangan, Executive chairman, Sri S.R Santhanam, Sri M. Mahadevan were on the dais with the Executive chairman welcoming the gathering, General Secretary presenting the annual report and the president addressing on the state of affairs in maths. education.

   Then followed a Special Lecture by the President, who engaged the audience on ramifications of π with active involvement of children present. Prof. K. Kameshwar Rao of RIE, Bhopal delivered, Prof. A. Narasinga Rao Memorial Lecture with Sri R. Athmaraman as Chairman, when lunch was announced.

   In the post lunch session paper presentation by adults followed by students took place. Sri S.R. Santhanam was totally involved in these sessions as in earlier years from scrutinizing, selecting and guiding delegates while presenting papers. After that delegates went for tea and visited the exhibition specially got up by the students of the host school.

   The second day started with written quiz with more than 100 participants writing with interest. This was followed by Prof. P.L Bhatnagar Memorial Lecture delivered by Prof. J. Pandurangan, the Executive Chairman, who filled ably, the void created by Prof. Deo Sarkar of Baramathi, with students taking interest more.

This was followed by a talk by Sri R. Athmaraman on the theme of the conference - **Diversification of Mathematics curriculum** - followed by paper presentation by adults when lunch was announced.

In the post lunch session Prof. S.C Agarkar delivered Prof. R.C Gupta Endowment Lecture on Bhaskaracharya 900. Sri S.R. Santhanam was in the chair. Then started the panel discussion on the theme with Prof. Agarkar as moderater with five ladies as panelists when tea was announced.

The General Body of the AMTI then met around 4.30pm with 39 members present after that an entertainment performance by the host school children took place for about an hour with aerobics by Boys and group dance by the girls, well received by the audience.

The Third day started with paper presentation by the students in two parallel sessions followed by tea.

Oral quiz with four teams of four each selected form written quiz was organized next with Sri R. Athmaraman as quiz master. He was ably helped by Sri. Lakshmi narayanan and Dr. Shanthi, when lunch was announced.

Due to shortage of time in the post lunch session only one delegate Sri M.S Solanki participated in short communication session.

There after Valedictory function started at 2pm with Sri Joshi, Minister for higher education, Madhya Pradesh as Chief Guest. Sri Rajesh Tiwari, Principal of the host school welcomed the gathering after prayer.

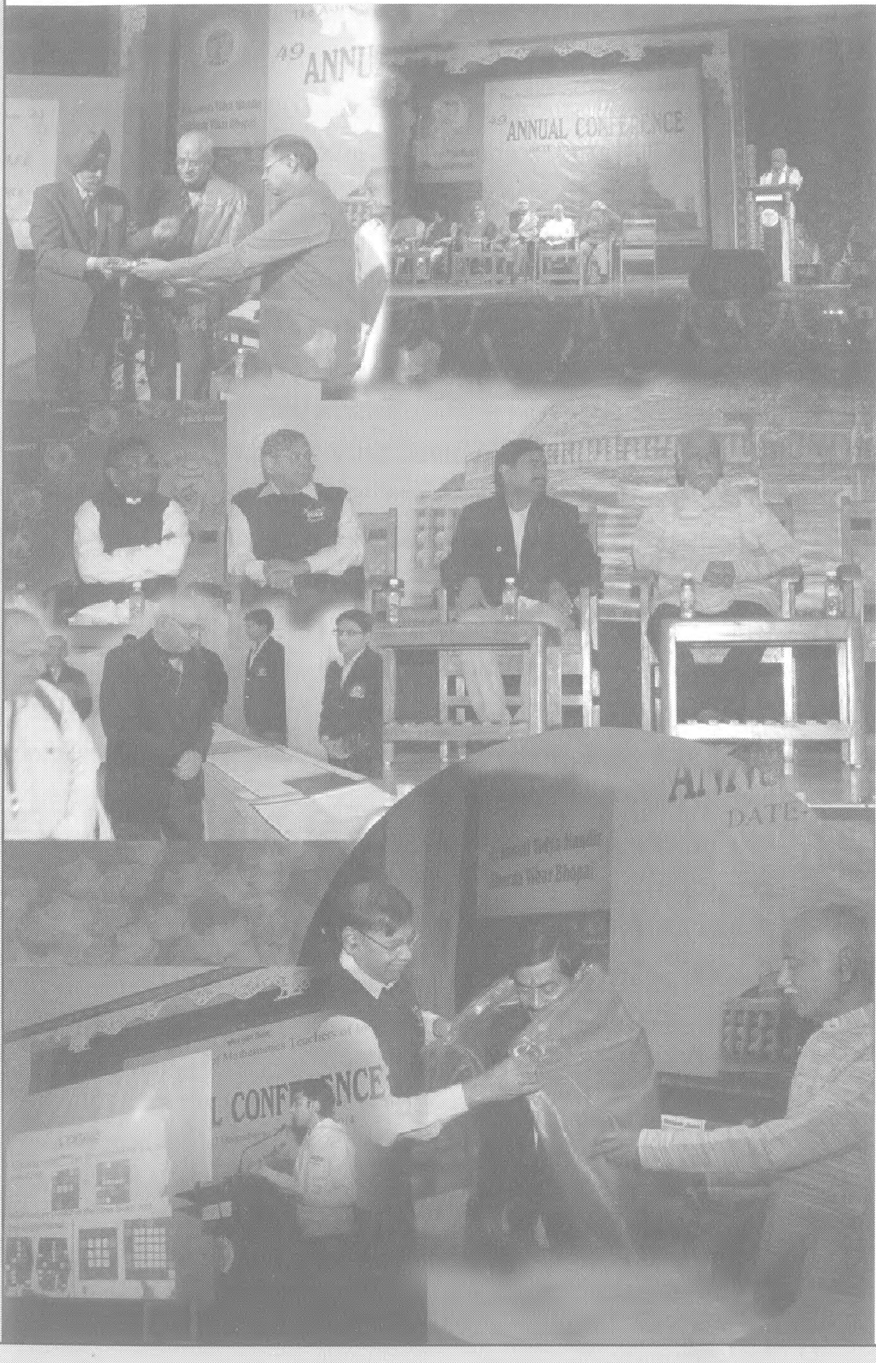
Prizes were distributed to the winners in the quiz competition. The Chief Guest congratulated the school and the AMTI for the initiative to develop math Education in Madhya Pradesh. The general secretary proposed vote of thanks specially requesting the chief guest to use his good offices to spread the service of AMTI all over the country including Madhya Pradesh. With national anthem the curtain came

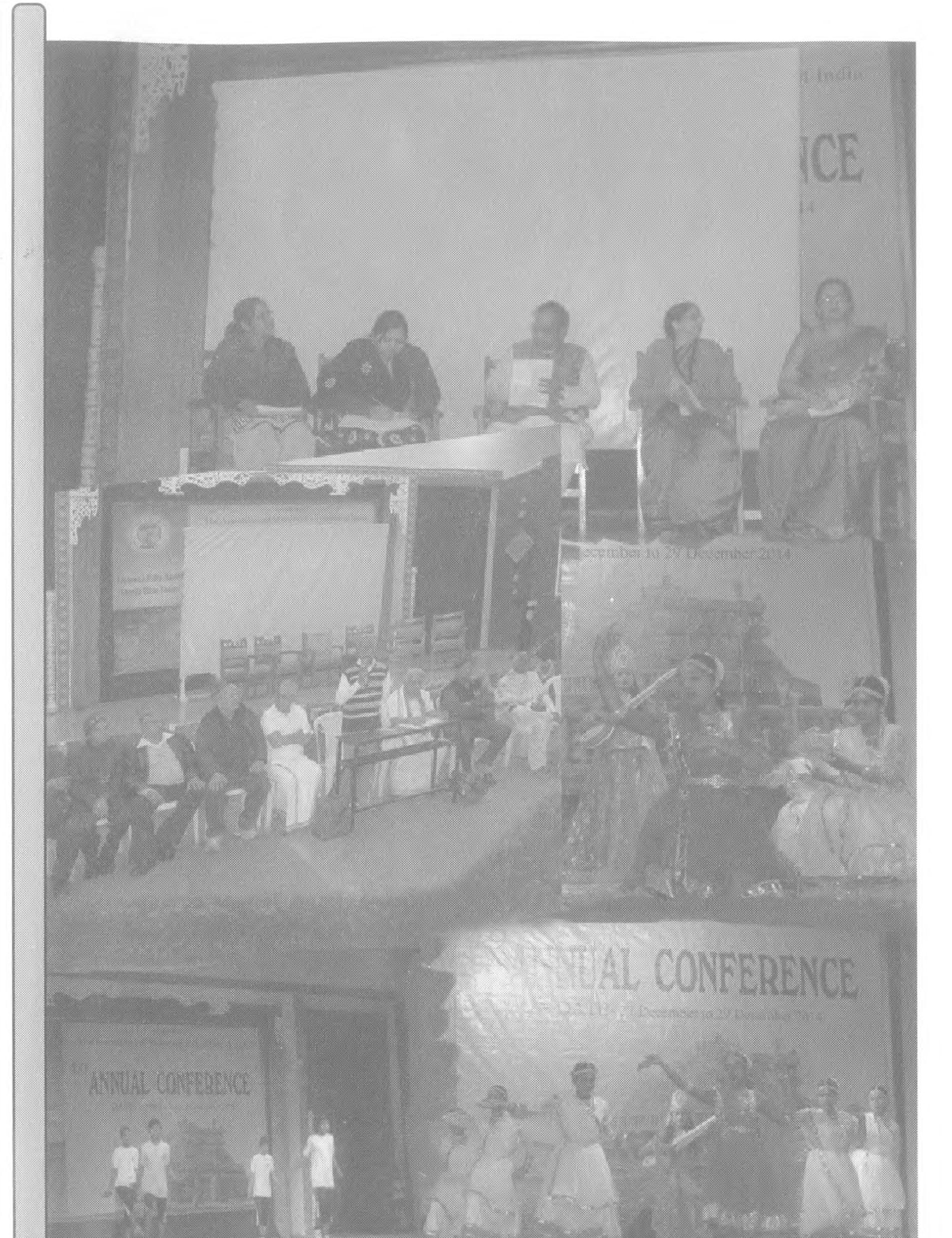
down on the 49th conference with delegates receiving certificates, souvenirs and packed dinner.

2. **GAT 2015** was conducted with 36 schools fielding 2708 children and the results finalized & announced with prizes and certificates.
3. **DST** Sponsored 4 day workshop was conducted at RV Teachers College Bangalore for Karnataka Teachers. Starting with 23 ending with 15 it went on well, orienting the teachers on innovative strategies, activities to introduce concepts, clarifying concepts and exploring solutions with their total involvement. Five resource persons helped in the same.
4. Following this two of our resource persons were invited to Arunachal Pradesh and Assam for the three 2-day workshops for teachers at Roing, Tezu and Dibrugarh with 30, 25 and 16 participants respectively. The participants assured to continue the work with enthusiasm in future.
5. With 31 students selected in **RMO eligible to write INMO,** three day workshop was organized for them on 29th to 31st January at IMSc, Chennai.
6. About **5600 books were received** form Sri P.K. Srinivasan's family to become part of AMTI library.
7. **New building** is nearly complete with 1+2 of floors in which already 1+1 in operation.
8. **Golden Jubilee** conference of AMTI will take place in December 2015.
9. Our **regular workshops** for Primary, Sub-Junior, and RMO group (Classes IX to XI) and teachers in May are announced in the website *amtionline.com* awaiting good response.
10. As announced in the earlier issue we are counting and nearing 50 -the number of workshops targeted, participated by one or more of our resource persons individually or in teams. The response and demands are encouraging indeed.

Some scenes from the conference and workshops one given in the following pages.

SCENES FROM 49 th CONFERENCE - BHOPAL-2014

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